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TRANSFORMS AND APPROXIMATIONS IN COST AND PRODUCTION FUNCTION RELATIONS

by

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Abstract

Known results in the mathematics of transform theory, e.g., as exhibited in Laplace transforms, are here used as a guide for exploring difficulties in what are sometimes called duality relations between cost and production functions in economic theory. Difficulties in specifying the set of production possibilities and in approximation with translog functions are identified by means of a simple "process analysis" example. Hazards associated with the possible uses of these ideas on energy studies and like topics are commented on in specific detail. Relations to another set of developments associated with a measure of decision making efficiency suggested by M. J. Farrell are also established and used to extend Shephard's lemma to non-differentiable polyhedral production possibility sets.

Introduction

Starting with the now classic works of R. W. Shephard, [1/] a long train of research efforts has been directed to the study of cost and production function relations. [2/] In this literature these relations are characterized as a "duality theory," possibly because what is involved is a pairing of points and hyperplanes as in some parts of classic geometry.[3/]

In this paper we propose to consider this "duality theory" and also its use, along with a variety of supposedly general forms of production functions, to deal with very important policy issues of the present day. Although our results are also applicable to other such forms as, e.g., the "generalized Leontief functions" of Diewert [5], we shall here focus on the so-called "translog function" for which properties like the following are claimed: [4/]

> "Our approach is [via the translog function] to represent the production frontier by functions that are quadratic in the logarithms of the quantities of input and output. These functions provide a local second order approximation to any [sic] production frontier."

And further,

> "For many of the production and price frontiers employed in econometric studies of production the translog frontiers provide accurate global [our emphasis] approximations."

1/ See [15] and [16].

2/ See, e.g., the recent survey by Diewert with accompanying discussions in [11].

3/ For a discussion of differences between this and the duality theory of mathematical programming, see [16] p. 11. See also Fenchel [7].

4/ [3] p. 28-29. See also [4].

Although the authors of the above quotations then observe that the "accuracy of the approximation must be determined separately for each application," they do not supply any guides or even any depictions (such as we shall provide) of the kinds of pitfalls that may be encountered.

Here we shall be concerned not only with the quality of these log-quadratic approximations but also with the a priori possibility of good approximations and determinations of the related functions that the "duality theorems" are supposed to yield.

In keeping with the way these topics are presently treated -- in both the theoretical and applied literature -- we shall deal with the "translog function" and this "duality theory" simultaneously. We shall, however, supply remarks for reader guidance at suitable points in the text, while trying to avoid a repetitious and unduly lengthy treatment of each of the separate (but related) topics. The central issue dealt with in this paper, as already noted, is, in any case, the use of the underlying duality theory in many phases of this work without reference to the fundamental hypotheses of its underlying mathematical structure and the possible pitfalls that may thereby be overlooked. Even though we shall proceed by simple examples (and counterexamples) we will nevertheless arrive at results which will provide needed illumination.

As a case in point we might mention the widespread use of the assumption that prices are constant and independent of the input and output quantities. I.e., as Shephard is himself concerned to note [1/]

[1/] Diewert et.al. [11] p. 206.

> "Throughout all of these duality theories (those described by Diewert 1/ and the writer 2/) the single most important assumption made limiting their usefulness is that prices ... for [the] input and output vectors are independent of their magnitudes...."

It is doubtful that this assumption reflects the actual behavior which underlies the data utilized in the studies we are referencing. Furthermore, attempts such as Griffin's [8] and [9] [3/] utilization of "pseudo data" [4/] generated via "process analysis" (and related) approaches also do not avoid this difficulty. There is the further difficulty that other underlying phenomena such as capacity limitations arising from resources, technology and other conditions may fail to coincide with the underlying hypotheses of the mathematical theory which contains the "duality theory" relationships being employed. [5/]

Although this paper will be concerned with mismatches that may be involved, as already noted, we should nevertheless also observe that we are faced with a real need for valid projections to enable us to deal with the possibilities of substantial alterations in prices, product mixes, and capacity limitations along with other economic and technological phenomena that are involved in the present problems of energy policy. Here, however, we want to focus on the former task only. I.e., we want to focus on shortcomings in present approaches. This will be done by some very simple

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1/ The reference is to the survey, chapter and the subsequent discussions in [11].

2/ I.e., Shephard himself.

3/ See also [10].

4/ This term appears to be due to L. R. Klein. See [9].

5/ E.g., as in Shephard's lemma (see below).

examples. First, however, we shall want to identify salient aspects of these "duality relations" with the mathematics of transform theory since known results in the relationships of the customary transforms of engineering and physics (e.g., Fourier and Laplace transforms) have suggested the kinds of possibilities that we shall show can also occur in these economic cost/production relations.

Transform Relations

For reasons of simplicity let us consider only the Laplace transform. First we recall that the Laplace transform F(p) of f(t) is given by

$$(1) \qquad F(p) = \int_0^\infty e^{-pt} f(t)\, dt$$

for suitable classes of functions.[1] There also exist various inversion transforms yielding f(t) from F(p), e.g.,[2]

$$(2) \qquad f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{tp} F(p)\, dp.$$

---

[1] Cf, e.g., D. V. Widder [17].

[2] Again cf. Widder [17] for details.

For cost and production theory there are the elegant relations, which we here propose to refer to as "Shephard transforms," 1/ as follows

$$\text{(3)} \qquad C(y,p) = \min.\ p^T x \quad \text{for} \quad \{x:\Psi(y,x)\geq 1\}$$

and

$$\text{(4)} \qquad \Psi(y,x) = \min.\ p^T x \quad \text{for} \quad \{p:C(y,p)\geq 1\}$$

Herein the set $\{x:\Psi(y,x)\geq 1\}$ is the production possibility set; 2/ i.e., it is an alternative characterization of the point-to-set production correspondence $y \longrightarrow L(y)$ between the output vector y and the set $L(y) \equiv \{x$: at least the output vector y is produced $\}$. This set is required to be non-empty and convex and, indeed, to be a cone, possibly truncated below, 3/ so that the components of the input vector $x\geq 0$ are always available in the requisite amounts for any $y\geq 0$ that may be stipulated. Similar remarks apply to the price vector p which, in transposed form, is used to generate the cost function $C(y,p)$ specified in (3) as well as the "distance function" $\Psi(y,x)$ specified in (4).

In employing e.g., pseudo-data methods, we are attempting to first obtain $C(y,p)$ from the equivalent characterization

$$\text{(5)} \qquad C(y,p) = \min.\ p^T x \quad \text{for} \quad x\varepsilon L(y).$$

Clearly the relationships between $C(y,p)$ and $\Psi(y,x)$ are similar to those of $f(t)$ and $F(p)$. One is the transform of the other in each pair. Moreover, with either of these pairs one may start with one of the functions to obtain the other. Each pair is related by the indicated operator.

1/ Cf. Fenchel [17] for geometric characterizations of this type of transform which might also be called "Shephard-Samuelson" transforms in view of their virtual simultaneous publication in [14]. See the discussion in [13].

2/ We are here conforming to the notation of Diewert *et al* as in [11].

3/ See Diewert [5].

Integration is the operation which effects the transformation in the Laplace pair and minimization is the operator for the Shephard pair. Finally, under suitable conditions there is a bi-unique correspondence between the function and its transform in each pair.

One of the well known relations, so-called "Tauberian Theorems", cf. [17], p. 192, Theorem 4.3, between F(p) and f(t) implies that if $F(p) \sim p^{-\nu}$, $\nu > 0$ as $p \to \infty$ then $t^{\nu - 1}/\Gamma(\nu)$ is asymptotic to f(t) as $t \to 0$. Thus if $\hat{F}(p)$ is asymptotic to F(p) in a neighborhood of ∞, it can generate an $\hat{f}(t)$ also asymptotic to f(t) as $t \to 0$. However, the inverse transform of $\hat{F}(p)$ may be nowhere close to f(t) for t substantially different from zero, and, a fortiori, there is no reason why $\hat{F}(p)$ need be close to F(p) away from the region of $p = \infty$.

For example, $f(t) = \begin{cases} \sin t, & t \geq 0 \\ 0, & t < 0 \end{cases}$ has Laplace transform $F(p) = (p^2+1)^{-1} = \sum_n (-1)^{n+1} p^{-2n}$. Thus, $\hat{F}(p) = p^{-2} - p^{-4}$ (approximates) is asymptotic to F(p) at $p = \infty$ (to the second order). Since $\Gamma(2) = 1$, $\hat{f}(t) = t - t^3/(3!)$ which is asymptotic to sin t as $t \to 0$. Clearly, $\hat{f}(t)$ is nothing like sin t elsewhere.

This behavior provides a clue to the likely existence of similar phenomena with the Shephard transforms. It is especially important to investigate these possibilities, as we shall now proceed to do, because of the wide-spread and increasing use of these techniques in areas like energy policy where exactly these kinds of difficulties can occur.

A Process Analysis Example

We now proceed by means of linear programming formulations of "process analysis" models.[1/] First, we note that a system so represented need not have constant returns to scale since, e.g., one may be employing

[1/] See, e.g., Markowitz and Manne [12].

piecewise linear convex functions in the constraints and functionals.[1]
Further, in such a formulation the input price may well vary with the level of the associated activities. Still further, the linear program may be employed algorithmically, e.g., as in integer programming, to solve quite arbitrary nonlinear problems.[2]

As we have already observed, it is essential in the Shephard transforms (between production functions and cost functions) that the input prices be constant and independent of the input quantities. This limitation of scope is similar to one also encountered in Laplace transform usages. For instance, the latter are useful for dealing with linear differential equations with constant coefficients to the extent of providing a general theory and approach to the solution of such equations. Other, more recondite types of differential equations, however, have no such transform theory for their treatment.

Most process analysis models involve restrictions such that not all y are possible. I.e., the set $L(y)$ may be empty for some output vectors y. Some input vectors may also be limited by capacities or other stipulations. While it may be possible formally to attribute infinite values to the cost function for unattainable y and, correspondingly, to attribute infinite values for Shephard's $\Psi(y,x)$ for unattainable values of x, no one seems to have dealt explicitly with the discontinuities involved. It is clear, however, that smooth, regular "everywhere defined" functions must fail to provide adequate approximations in many cases of great importance.

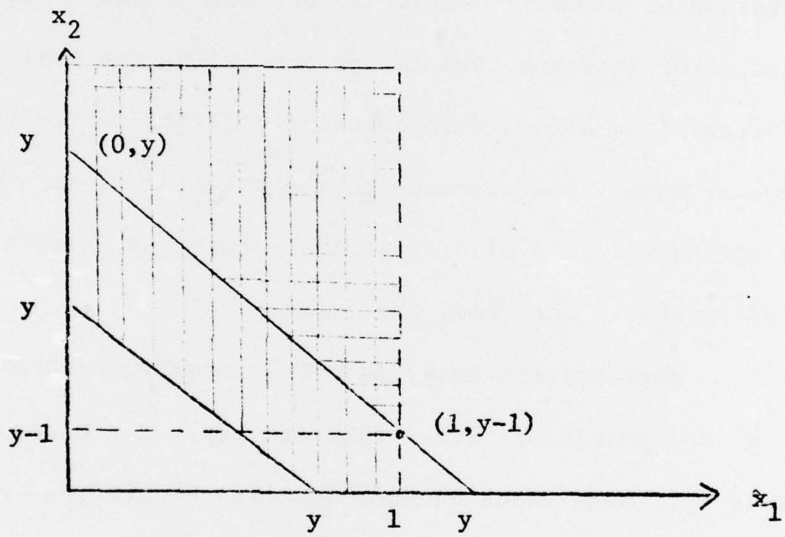
In order to make the preceding considerations more concrete we now proceed by means of a very simple example involving only one output and two inputs. Thus, let

$$(6)\qquad L(y) = \{(x_1,x_2):\ x_1+x_2\geq y,\ x_1\leq 1,\ x_1,\ x_2\geq 0\}.$$

1/ See, e.g., [1].

2/ See the discussion of "algorithmic completion of a model" in [1].

The set $L(y)$ is shown in the following Figure. [1/] The horizontal shadings represent $L(y)$ for $y>1$. The vertical shadings represent $L(y)$ for $y\leq 1$ Notice that in both cases there are only two "cost efficient" extreme points in the set. Thus in the determination of $C(y,p) = \min. p^T x$, $x \varepsilon L(y)$, where $y\geq 0$, one or the other of the extreme points gives the minimum, depending on the relative values of p_1 and p_2, the components of p associated with x_1 and x_2 respectively.



We can therefore represent the values of $C(y,p)$ in a two way table as follows:

(7) $C(y,p)$ =

	$p_1\leq p_2$	$p_1>p_2$
$y\leq 1$	$p_1 y$	$p_2 y$
$y>1$	$p_1+p_2(y-1)$	$p_2 y$

[1/] Notice that $L(y)$ is not a truncated (below) cone.

Note that in the general case of a polyhedral set of production possibilities involving n input variables x_j and m output variables y_i we could, <u>in principle</u>, construct a larger table with one column for each "cost efficient" production possibility and with one row for each facet in the y_i for which different extreme points could be designated. At any rate, as this example already makes clear, there can be different functional forms for $C(y,p)$ corresponding to the various (y,p) possibilities.

We now develop additional tables of the same type to exhibit some other relevant properties. For instance, taking natural logarithms in the above table yields

(8) $\ln C$:

$\ln p_1+\ln y$	$\ln p_2+\ln y$
$\ln[p_1+p_2(y-1)]$	$\ln p_2+\ln y$

Next, the partial elasticities with respect to input price are respectively exhibited via

(9)

$\dfrac{\partial \ln C}{\partial \ln p_1}$:

1	0
$\dfrac{1}{p_1+p_2(y-1)}$	0

$\dfrac{\partial \ln C}{\partial \ln p_2}$:

0	1
$\dfrac{y-1}{p_1+p_2(y-1)}$	1

Finally, we record the efficient input demands x_1^*, x_2^*, which, according to Shephard's lemma, are given by the partial derivatives

$\frac{\partial C}{\partial p_1}$, $\frac{\partial C}{\partial p_2}$:

$$
\text{(10)} \qquad
\begin{aligned}
x_1^* = \frac{\partial C}{\partial p_1} \quad &: \quad \begin{array}{|c|c|} \hline y & 0 \\ \hline 1 & 0 \\ \hline \end{array} \\[2em]
x_2^* = \frac{\partial C}{\partial p_2} \quad &: \quad \begin{array}{|c|c|} \hline 0 & y \\ \hline y-1 & y \\ \hline \end{array}
\end{aligned}
$$

At this point we should remark that there is another important stream of production function research stemming from constructs first formulated by M. J. Farrell. 1/ As we have elsewhere shown, 2/ this work can be made to rest on the duality theory of mathematical programming in ways which avoid the problems and pitfalls that we shall now examine. With this in mind we shall therefore shortly supply a generalization of Shephard's lemma which makes it applicable to polyhedral production possibility sets. In this extended form, as we shall see, Shephard's $\Psi(y,x)$ can be shown to be the reciprocal of Farrell's measure of technical efficiency and so possible further relations from this quarter may also need to be considered.

1/ See [6] and [2].

2/ See precise linear-nonlinear programming characterization given in [2].

Translog Illustration

Now suppose we approximate $C(y,p)$ to the second order in the logarithms of p as in a translog function. For simplicity of notation, we do not here bother with all of the corresponding approximation in y since taking derivatives with respect to p_1 and p_2 will remove the terms involving only y. Thus, for the translog function approximation, we write

$$
\begin{aligned}
\ln\tilde{C} &= a_{10} \ln p_1 + a_{02} \ln p_2 + a_{11} (\ln p_1)^2 \\
&+ a_{22} (\ln p_2)^2 + a_{12} \ln p_1 \ln p_2 + a_{13} \ln p_1 \ln y + a_{23} \ln p_2 \ln y \qquad (11) \\
&+ \text{terms in logarithms of } y.
\end{aligned}
$$

Thus

$$
\frac{\partial \ln\tilde{C}}{\partial \ln p_1} = a_{10} + 2a_{11} \ln p_1 + a_{12} \ln p_2 + a_{13} \ln y \qquad (12)
$$

compares to the first expression in (9) -- <u>viz</u>.,

1	0
$\frac{1}{p_1 + p_2(y-1)}$	0

and

$$
\frac{\partial \ln\tilde{C}}{\partial \ln p_2} = a_{02} + 2a_{22} \ln p_2 + a_{12} \ln p_1 + a_{23} \ln y \qquad (13)
$$

compares to the second expression in (9) -- viz.,

0	1
$\frac{y-1}{p_1+p_2(y-1)}$	1

There is no way of choosing the values of a_{ij} to get a good match in more than two regions for either one of these two elasticities. E.g., suppose we approximate $\frac{\partial \ln C}{\partial \ln p_1}$ by $\frac{\partial \ln \tilde{C}}{\partial \ln p_1}$ in the neighborhood of $p_1=p_2=1$ and $y=1$, where $\ln p_1=\ln p_2=0$ and $\ln y=0$. Then for a good approximation with $p_1<p_2$, $y\leq 1$ in this neighborhood we must have $a_{10}\simeq 1$. But for $y\leq 1$, $p_1>p_2$ in this neighborhood we need $a_{10}\simeq 0$. Both values cannot be simultaneously assigned to a_{10}. Similar remarks apply to the other elasticity and to other neighborhoods.

Finally, we turn to the efficient inputs obtained from Shephard's lemma for further comparisons via

$$(14) \qquad \tilde{x}_1^* = \frac{\partial \tilde{C}}{\partial p_1} = \frac{\tilde{C}}{p_1}\frac{\partial \ln \tilde{C}}{\partial \ln p_1} = \frac{\tilde{C}}{p_1}\left[a_{10} + 2a_{11}\ln p_1 + a_{12}\ln p_2 + a_{13}\ln y\right]$$

instead of the first expression in (10) -- i.e.,

$x_1^* =$

y	0
1	0

Similarly,

$$(15) \qquad \tilde{x}_2^* = \frac{\partial C}{\partial p_2} = \frac{\tilde{C}}{p_2}\frac{\partial \ln \tilde{C}}{\partial \ln p_2} = \frac{\tilde{C}}{p_2}\left[a_{02} + 2a_{22}\ln p_2 + a_{12}\ln p_1 + a_{23}\ln y\right]$$

instead of the second expression -- i.e.,

$x_2^* =$

0	y
[illegible]	[illegible]

If, as before, we focus on approximation in the neighborhood of $y=1$, $p_1=p_2=1$ (hence $\ln p_1=\ln p_2=0=\ln y$) then $\tilde{C}/p_1 \approx \tilde{C}/p_2 \approx 1$. See (11). Thus in (14) and (15) the values of a_{10} and a_{02}, play the same roles as before with analogous consequences and evidently similar conclusions as in the preceding case again apply.

The above very simple examples thus provide what is wanted. These difficulties are not matters which can be handled by simply adding terms as in a power series representation. The failure is in the functional forms <u>per se</u> and hence is a failure of the methodology itself.

To the argument that the translog function is intended as a good approximation to sufficiently differentiable functions (which are otherwise arbitrary) we now enter two remarks as follows. First, as our simple example shows, process analysis models will generally yield only piecewise differentiable functions. Second, a local second order approximation by a translog function to a twice differentiable function may be unsatisfactory even when the latter is infinitely differentiable. 1/ A simple example involving only one variable is as follows

$$\ln C = a_{00} + a_{10}\ln p_1 + a_{11}(\ln p_1)^2 + f(p_1),$$

(16) where

$$f(p_1) = \begin{cases} 0 & , \ 0 \le p_1 \le 1 \\ e^{-(p_1-1)^{-2}} \ e^{p_1^4} & , \quad p_1 > 1. \end{cases}$$

Thus, in the neighborhood of $p_1=1$, C is closely approximated by $a_{00} + a_{10}\ln p_1 + a_{11}(\ln p_1)^2$. As p_1 increases beyond 1, however, $f(p_1)$ increases as p_1^4 power of e and cannot possibly be overtaken by logarithmic terms. 2/

1/ In their original publication [4] Professors Christensen, Jorgenson and Lau assert that the translog function provides a second order approximation to an "arbitrary" functional form but their subsequent usage makes it clear that they are restricting themselves to differentiable functions with a sufficient number of derivatives.

2/ Note that $f(p_1)$ is a classic example of an infinitely differentiable function

Production Possibility Sets by Shephard Transform

We now return to the pure theory of production, i.e., the theory as Shephard developed it, [1/] without further explicit reference to the translog function. Having obtained $C(y,p)$ from the "process analysis" example characterizing $L(y)$, we proceed to calculate $\Psi(y,x)$ as the Shephard transform of $C(y,p)$, which is given in (7).

The Shephard transform is

$$\text{(17)} \qquad \Psi(y,x) = \inf.\ p^T x \text{ for } p \geq 0 \text{ and } C(y,p) \geq 1.$$

For $C(y,p)$ as in (7), this becomes a linear programming problem for which we need only examine and compare the values of $p^T x$ at extreme points.

For example, for $y \leq 1$, $x > 0$ and $p_1 \leq p_2$, we have from (7) that $p_1 y \geq 1$. Hence for a minimum $p_1 y = 1$, or $p_1 = 1/y$. Also $p_1 \leq p_2$ implies for a minimum that $p_2 = p_1$. Therefore, the minimum of $p_1 x_1 + p_2 x_2$ for $p_1 \leq p_2$ is given by $(x_1 + x_2)/y$. If instead $p_1 > p_2$, this same argument interchanges p_1 and p_2 and we get the same result -- i.e., $(x_1 + x_2)/y$. Thus for $y \leq 1$, $\Psi(y,x) = (x_1 + x_2)/y$.

The reasoning becomes more involved for $y > 1$, but reduces to consideration of only two cases, namely,

$$\text{(18)} \qquad x_2 \leq x_1(y-1) \text{ and } x_2 > x_1(y-1).$$

Finally, we obtain for $\Psi(y,x)$ the piecewise representation,

---

1/ See the introduction in [16].

(19) $\Psi(y,x)$:

	$x_2 \leq x_1(y-1)$	$x_2 > x_1(y-1)$
$y \leq 1$	$(x_1+x_2)/y$	$(x_1+x_2)/y$
$y > 1$	$x_2/(y-1)$	$(x_1+x_2)/y$

It may be noted that the production possibility set (for any fixed y) given by $\Psi(y,x) \geq 1$ is a truncated (below) cone in contrast to L(y), which is not. However, the so-called efficiency frontier is the *same* for these two different production possibility sets.

Thus, if the production possibility sets are derived from C(y,p) as $\{(y,x): \Psi(y,x) \geq 1, x,y \geq 0\}$ from the Shephard transform, they will always be truncated (below) cones and will fail to give information about limitations of the production possibility sets -- which may be of critical importance for policy issues such as are now being examined by means of this transform theory.

In fairness to Shephard, we should note that he explicitly states that he is avoiding attending to such constraints in order to deal with "the unconstrained possibilities of a technology." Indeed, as he observes,[1/]

> "No limitations will be put upon the available amounts of the factors of production, because this implies reference to some particular production unit which confounds the notion of a production function with some implicit economic decisions or production plan, the variety of which is unlimited, preventing a clear, unambiguous and generally applicable definition of the production function."

So long as one is concerned with small "in the neighborhood variations" or even if one undertakes extensive variations for the purpose of better understanding the behavior of the function within the present neighborhood

---

1/ See [16] p. 4.

this is not objectionable and may even be desirable. With the kinds of large-scale adjustments and the kinds of constraining relations that require attention for present energy problems, however, this approach becomes questionable. These questions become even more serious with other assumptions such as are embodied in the kinds of price-quantity relations which we remarked on earlier -- e.g., the assumption that prices are fixed at constant values independent of the input quantities being considered.

To conclude this section, it should be obvious that the same efficiency frontier can hold for any number of different production possibility sets, and not just the two we have exhibited. There is, therefore, ä priori no way of uniquely determining the "correct" production possibility set starting with a $C(y,p)$. The fact that the Shephard transform $\Psi(y,x)$ will give back the starting $C(y,p)$ -- as is readily verified for our example -- has no bearing on the situation. Thus, methods which propose to start with a "fairly arbitrary" form (e.g., specific, except for parameters to be determined) cannot, ä priori, necessarily give us a correct production possibility set.

Extension of Shephard's Lemma

The purpose of the above discussion is to open rather than close issues like the above for research, especially now when these kinds of functions and related approximations are being applied to important issues of national policy. We have elsewhere [2] suggested how the developments in the research on these transformation relations can be brought into contact with another series of researches in production function estimation. It may therefore be useful to close on a further note in the latter vein by relating Farrell's concept of technical efficiency [6] to Shephard's $\Psi(y,x)$.

In [6] Farrell considered the case of a collection of n firms with input vector P_j and output vector Q_j for the jth firm.

In this same notation Shephard's correspondence can be represented

$$(15)\qquad \begin{aligned} y &\longrightarrow L(y) = \{x:\ x = \sum_j P_j\lambda_j \ \text{ for } \ \sum_j Q_j\lambda_j \geq y,\ \lambda_j \geq 0\} \\ x &\longrightarrow P(x) = \{y:\ y = \sum_j Q_j\lambda_j \ \text{ for } \ \sum_j P_j\lambda_j \leq x,\ \lambda_j \geq 0\}. \end{aligned}$$

Next Shephard defines

$$(16)\qquad \Psi(y,x) \equiv \frac{1}{\nu^*(y,x)}\ ,\ \text{where}\ \ \nu^*(y,x) = \min.\nu,\ \nu x \varepsilon L(y),\ \nu \geq 0.$$

But, as we have elsewhere shown, $\nu^*(y,x)$ = Farrell's technical efficiency for a firm with input x and output y. Thus, the problem min.ν, $\nu x \varepsilon L(y)$ is equivalent to the linear programming problem with <u>fixed</u> x and y,

$$(17)\qquad \begin{aligned} &\min.\nu \\ &\text{with:}\ \ \nu x - \sum_j P_j\lambda_j = 0 \\ &\qquad\qquad \sum_j Q_j\lambda_j \geq y \\ &\qquad\qquad \nu,\ \lambda_j \geq 0\ . \end{aligned}$$

Next, we obtain a nonlinear programming problem for C(y,p), x is now variable,

$$(18)\qquad \begin{aligned} C(y,p) &= \min_x\ \{p^T x : \Psi(y,x) \geq 1\} \\ &= \min_x\ \{p^T x : \nu^*(y,x) \leq 1\} \\ &= \min\ p^T x \\ &\quad \text{with:}\ \ \nu x - \sum_j P_j\lambda_j = 0 \\ &\qquad\qquad \sum_j Q_j\lambda_j \geq y \\ &\qquad\qquad\qquad \nu \leq 1 \end{aligned}$$

Or

$$C(y,p) = \min \frac{1}{\nu} \sum_j p^T P_j \lambda_j$$

(19) subject to $\sum_j Q_j \lambda_j \geq y$

$$\nu \leq 1$$

$$\nu , \lambda_j \geq 0.$$

Thus, since the minimum occurs for $\nu=1$ we return to a linear programming problem

$$C(y,p) = \min \sum_j p^T P_j \lambda_j$$

with: $\sum_j Q_j \lambda_j \geq y$

$$\lambda_j \geq 0.$$

If λ^* is optimal, then

$$C(y,p) = p^T \sum_j P_j \lambda^*_j$$

But $\sum_j P_j \lambda_j^* = x^*$. So $C(y,p) = p^T x^* = \sum_i p_i x_i^*$.

Thus, we have obtained a generalization of Shephard's lemma to this non-differentiable polyhedral production case, namely, <u>formally</u>

$$x_i^* = \frac{\partial C}{\partial p_i} .$$

Of course, this derivation need never be performed if we develop $C(y,p)$ by solution of this LP problem since

$$x^* = \sum_j P_j \lambda_j^* ,$$

as was noted in the course of the above proof.

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13. ABSTRACT

Known results in the mathematics of transform theory, e.g., as exhibited in Laplace transforms, are here used as a guide for exploring difficulties in what are sometimes called duality relations between cost and production functions in economic theory. Difficulties in specifying the set of production possibilities and in approximation with translog functions are identified by means of a simple "process analysis" example. Hazards associated with the possible uses of these ideas on energy studies and like topics are commented on in specific detail. Relations to another set of developments associated with a measure of decision making efficiency suggested by M. J. Farrell are also established and used to extend Shephard's lemma to non-differentiable polyhedral production possibility sets.

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